An Investigation into the Modelling of Word Problems

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The main aim of the present study, using first year university students as subjects, was to discover whether there are any differences in the success rates of students between problems in which they just translate the word problem into an algebraic equation and problems in which they are asked to solve the equation as well. Although the success rates were not very different, a large proportion of the correct solutions to the equation were obtained without using an algebraic equation as part of the solution process. A secondary aim of the study, motivated by the Student/Professor problem, was to investigate students' errors when the relationship between two variables was given as a quotient instead of a product. Because this study raised further questions, it was repeated with some modifications to the questions. This provided more detailed information about students' understanding of variables and showed a significant difference between the proportions correct depending on the order in which the questions were presented.

The errors students make when attempting to write simple linear algebraic equations have been studied for many years and by a large number of researchers, an early example being Paige and Simon (1966). Nevertheless some of the mechanisms involved in the translation process, which cause the errors, remain elusive. Of course the modelling process depends on an understanding of the meaning of algebraic variables and this has also been a focus of much research, for example Kuchemann (1978). More recently MacGregor and Stacey (1997) have shown that students' misunderstandings about the meaning of algebraic symbols, depends not only on cognitive development but on various environmental factors, one being the type of instruction used to introduce algebraic symbols.

Two processes for translating word problems into algebraic equations, which lead to errors, syntactic translation and static comparison were identified by Clement (1982).
The process of syntactic translation, where students use the word order in the story to form the equation, frequently results in the relationship between the two variables being reversed, as in the now famous students and professors problem,

> Write an equation using the variables S and P to represent the following statement. "There are six times as many students as professors at this university." Write S for the number of students and P for the number of professors.

The answer, $6S = P$ is claimed to be an example of syntactic translation because the word "six" directly precedes the word "students". On the other hand, static comparison produces the result $6S = P$ because there are more students and the number six expresses this fact. However Macgregor and Stacey (1993) showed very convincingly, by using questions in which syntactic translation would have produced the correct equation, that many students still reversed the variables in their responses. They therefore concluded that the cognitive models that students use to write mathematical equations are not linear but very complex and that a great deal more research is required on how students use these models to construct mathematical equations and how they can be helped to develop the thought processes required to translate word problems into equations correctly.

The present study further investigates some aspects of this problem. It compares students' responses to two problems, which have similar syntax, it discusses the importance of the particular letters used in the statement of the problem and investigates the methods students use to solve an equation. Because most students complied with the request to explain their responses, a number of other issues emerged from an analysis of their responses, which were not a part of the original design. In studies where students just write an answer or select one from the answers provided this additional information is not available. More detailed information then needs to be obtained from interviews.

In the repeat study, the syntax of the first two questions was altered and this gave more detailed information about students' understanding of variables and also confirmed the results in the first study.

Even though the research into the modelling and solution of simple algebraic equations has produced results that could well lead to improved teaching strategies this does not appear to have occurred. The problems identified twenty years ago are still there, both at school and undergraduate level.

Research Questions

1. Are students more successful at translating a statement given in words that leads to an expression or to an equation?
2. How do students respond to translating a statement involving a quotient rather than a product?
3. How are students' responses to modelling word problems affected when a numerical solution is asked for?
4. How does students' understanding of variables affect their ability to model word problems?
5. Can the correct modelling of word problems be affected by something as simple as the order in which the problems are presented?

Method

The students in this study were two classes of first year engineering students, in consecutive years, at the Queensland University of Technology. There are two first year mathematics courses for engineering students. In the first year of the study, one class of 243 students, consisted of those who were considered to be well prepared for university mathematics and one class of 232 students consisted of those who were less well prepared but who had studied mathematics to the end of secondary school. The second group were chosen for this study. All these students would have been at least 17 years old, many considerably older. In the second year of the study the class size was larger and the less well-prepared group were targeted again.

In the initial study, the students were asked to answer the following three questions during a lecture in their second week at university.

1. In a QUT classroom, there are 2 chairs beside each table. If there are n chairs, how many chairs and tables are there altogether?
2. In an Engineering Maths test, the number of students who pass, is 3 times the number of students who fail. If the number of students who pass is n, write an equation for t, the total number of students in the class.
3. In an Engineering tutorial, there are twice as many females as males. (A very unusual occurrence.) If there are 33 students in the class and x of them are females, how many males are there in this class?

Technically, all the letters required for making correct responses are provided, but students were asked to label any new letters they wished to introduce. As stated, the questions all involve quotients rather than the products used by most previous researchers. This was intended to make the questions a little more difficult because the sample consisted of university, rather than junior secondary school students. In several pilot studies, questions of varying difficulty were tried but it became evident that if the questions were more difficult than those above, a large proportion of students would simply make no attempt to respond. The questions were not part of any formal assessment so the students had to feel that they were able to make a response without being coerced into doing so. In all, 217 responses were obtained, which it is reasonable to assume, was the number of students who were present at that particular session.

In the first and second questions, the syntax is very similar but the first asks for an expression while the second asks for an equation connecting two variables. Neither of these questions provides enough information for obtaining a numerical solution. The third question provides more information and does ask for a solution. There was a large space below each question and students were asked to write any working, explanation and thinking steps.

In the repeat study, the questions were presented in the same format and under the same conditions but the wording was changed as follows:

1. In a QUT classroom, there are 2 chairs beside each table. If there are n chairs, how many chairs and tables are there altogether?
2. In an Engineering Maths test, the number of students who pass is 3 times the number of students who fail. If the number of students who pass is p, how many students are in this class?
3. n coins are to be divided between Jack and Jill. Jill must first get k coins. The rest of the coins are divided so that for every x coins that Jill gets, Jack gets z coins.

   How many coins does Jack get?

   How many coins does Jill get?

The first two questions now both ask for an expression and are worded very similarly. Approximately half the students had the questions in this order. For the remainder the order of the first two questions were reversed. The third question was included to show what happens when the modelling becomes more difficult.

Results

Variables

Of the 217 responses, 56 answered all three questions correctly and were excluded from most of the remainder of analysis. The questions were clearly too easy to challenge these students. These 56 students were only included in the part of the study which looked at students' use of their own variables. The questions themselves contained all the variables necessary to write the response and these variables did not use the first letters of any of the words in the problem. Table 1. shows the proportions of students who introduced variables which were the first letters of the objects being counted in at least one question. If students introduced another variable such as y, which was not the first letter of a key word, this was ignored.

Table1

Use of Variable Names

	All 3 correct (N =56)		Not all correct (N=161)		Total	
Initial letter as	No.	%	No	%	No	%
variable name	16	29	75	47	91	42

Sometimes the student did not introduce a new variable but felt the need to rename an existing variable, to reflect the object it was counting, for example in Q3. $x = f$, in other cases the new variables were introduced for the other objects named in the questions, tables, failing students and males. The table suggests that the weaker students, those that did not get all three questions correct, tended to need letters that represented "things" more than the better students. The use of letters that bore no visible relationship to the objects they were supposed to be counting, appeared to add another level of abstraction to the problem. Most students complied with the instruction to label any new variables they wished to use but statements like "t = tables, c = chairs" were very common and in some cases that is exactly what the students meant because such statements were then followed by formulae such as □ or even $A = (2c + 1t)^n$. This type of error, where n is used as an adjective, qualifying the name of the object being counted, occurred 13 times. This error has been observed in other studies, for example Stacey and MacGregor (1997) observed that this phenomenon persisted when junior secondary students were tested in three years. These results show that for some students this misconception persists much longer.

Expressions Versus Equations

The only essential difference between questions 1. and 2., is that the first question asks the student to write an expression while the second asks for an equation connecting two variables.

Mathematically an expression is a simpler object than an equation because no relationship is implied and an understanding of equality is not required. However students are clearly accustomed to being required to write equations so many simply inserted the letter t and an equal sign. Provided the expression was correct, this was regarded as a correct response. The success rates of students in these two questions are shown in Table 2.

Table 2

Correct Responses for Expressions and Equations

Question 1(Expression)	Question 2 (Equation)
Correct	Correct
44%	60%

This is the opposite of the expected result and one explanation is that in local beginning algebra textbooks, students' first encounter with algebraic symbols is with writing functional relationships. This is followed later with modelling simple equations, so they may have had no experience in writing expressions. Another explanation emerged from the repeat study and will be described later.

Inconsistency

The wording of the first two questions was intended to be very similar, with just the symbol for total absent in Q1. This enabled a comparison to be made between students who modelled both questions as quotients, both as products or who wrote one of each. Table 3. shows the frequencies of these pairs of responses.

Table 3

Choice of Products and Quotients

	Both products	Both quotients	Mixed	Unclassified
Number	24	80	42	15
Percentage	15	50	26	9

A response was labelled a product if it had the form □or $3n + f = t$ or if it was not quite complete but the product form was clear. A typical quotient response was
□ or f = no. who failed, $t = 3f + f$, where the quotient has been turned into a product but is a correct translation of the information. The unclassified responses were the result of both questions not being answered, of equations involving other operations such as differences or powers or illegible responses. Such a large number of inconsistent responses suggests that some students have conflicting processes, whether they be syntactic translation or static comparison or some other procedure, for dealing with this kind of problem and that the particular response that is made depends on which process gets the upper hand at any particular moment. This kind of vacillation was evident in the student interviews reported by Clement (1982) in connection with the students/professors problem. Students might write the correct equation, then decide it was not correct and replace it with an incorrect one or vice versa. This was also happening in some of my written responses where students wrote an equation, crossed it out and replaced it with a different one, usually quotient $\rightarrow$ product or product $\rightarrow$ quotient. In these cases the student's final uncrossed-out response was used but it was clearly not very reliable.

Solving the Equation

When a problem requires a numerical solution, students focus almost exclusively on "getting the right answer". This is what has been emphasized in their mathematical upbringing from an early age and what is learned early in life is difficult to alter. As a result, when confronted by Q3. a large proportion of students abandoned algebraic symbols, which they had used in the first two questions, and concentrated on producing an answer. Table 4. summarizes these results.

Table 4

Writing Algebraic Equations

	No attempt to write an equation	Wrote equation with an incorrect answer	Wrote equation and a correct answer	Unclassified
Number	59	23	57	22
Percentage	37	14	35	14

Most of the unclassified responses did not answer this question. Most of the students who wrote equations, introduced new variables, and at least initially, wrote an equation with two variables. The three equally popular variable pairs were *x* and *m, x* and *y*, or *m* and *f*. For some of these students the learned relationship between *x* and *y* took priority over the context related *m* and *f*. The reasons why some equations did not lead to a correct solution were that:

- once a second variable was introduced, the student could not eliminate it
- the equation contained a fraction, $x/2$ and the student could not solve such an equation
- the symbols representing the number of males and the number of females were used inconsistently.

Those students who did not attempt to write an equation, and showed their reasoning, used proportion, mostly successfully. Interestingly, when using proportion, the students did not reverse the variables, whereas those who attempted to write equations did so frequently, as in the first two questions.

Results from the Repeat Study

In the repeat study, the third, difficult question was only answered correctly by 8 of the 250 students who responded. 38 students made no response at all to this question and of the remainder most only managed to correctly calculate that there were $n - k$ coins left to share. This validates the original decision to keep the questions simple in order to obtain analysable responses.

Question Order

With the syntax in the first two questions now so similar, there did not appear to be any logical reason why question order should influence the proportions of the two questions that were answered correctly. However question order was a significant effect as shown in Table 5. For each question, the difference between the proportions was significant with $p < 0.01$. Furthermore the classroom question had a higher proportion of correct responses when it was the second question, while the students' question had a higher proportion of correct responses when it was the first question. This difference may be partly explained by an analysis of the students' failure to add the two terms in the expression.

The reason why many students produced incorrect answers, was because they did not add the number of chairs to the number of tables or the number of failing students to the number of passing students. In fact, of the 107 students who did not answer the first two questions correctly, 13% did not add in both questions, 12% did not add in the students question and 24% did not add in the classroom question.

Table 5
Percentage of Correct Responses by Position on Question Paper

	Position 1		Position 2	
	% Correct	Total	% Correct	Total
Classroom Question	0.53	137	0.71	113
Students Question	0.83	113	0.69	137

A few students made it clear that you could not add chairs and tables and this may have troubled other students who did not explain their answer. In the students question, they were more comfortable adding two different kinds of *students*, those who passed and those who failed. Only a small number of students showed explicitly that they thought that the letters represented "things", as in the first study, but those students who couldn't add tables and chairs were probably also thinking about the things and not the numbers of things. This may also explain the discrepancy between the percentages correct in Table 2.

Discussion

It is clear that before students can successfully translate word problems into expressions or equations they first need to have a thorough grasp of the meaning of algebraic symbols. The fact that 13 students demonstrated that for them, the letters represent things and that 91 out of the whole group of 217, or 42% needed to use the initial letters of words as an aid to forming equations, suggests that their understanding of algebraic symbols is not very thorough. This translation of symbols is frequently seen in reverse when students learn a mathematical procedure such as finding a derivative. If the function to be differentiated contains a variable other than x, then students whose understanding of the procedure is not very robust, change the variable to x first, and then apply the procedure.

The reason why students write an equation when the context only requires an expression, is probably because they have only encountered the translation into symbols in the context of equations. An examination of commonly used local school textbooks for the year in which algebra is introduced, shows that they do provide a few exercises in which students are asked to construct expressions from words or diagrams. However they then move on to looking at algebra in different ways, without making any attempt to integrate all these different approaches. The idea of translating words into formulas may never be revisited. Furthermore, the chapters on algebra are interspersed with chapters on geometry and various arithmetic techniques, so that if the sequence in the textbook is followed in instruction, it is difficult for students to make the necessary connections between concepts.

Although not originally research questions, the results showing the inconsistency of students' responses suggest that some of the hierarchies of understanding the meaning of variables and on reversal when writing equations, may not be as clear-cut as they first appeared. These students have had numerous mathematical experiences during their schooling and their problem appears to be that these experiences have not developed into a consistent cognitive structure. There is a conflict in students' minds between different interpretations and at any moment the answer to any single item is somewhat arbitrary.

The unexpected significance of the order in which the questions were presented may be partly explained by the fact that if the students question was first and the respondent added the two terms, then that may have influenced their decision to add in the classroom question. Alternatively if the classroom question was first and they thought that adding was not appropriate that may have carried over to the students question. In any case the failure to add suggests that many students are still thinking of variables as things, not numbers. It also demonstrates that what may appear as insignificant differences in wording to a mathematician are significant for students. "how many students are in this class" and "how many chairs and tables are there altogether" do not convey the same message.

Question 3. did not specifically ask students to use algebra to obtain a solution but it did include a variable and was preceded by two questions where algebraic symbols were necessary. So 49% followed these hints and used algebra but a surprising 37% chose to ignore the hints and go straight to an arithmetic procedure. A small number of students used arithmetic and then attempted to justify the solution algebraically. Again there is evidence of a conflict: do what the teacher wants and write an equation versus get the answer in the simplest possible way. Some students, who answered the first two questions correctly, wrote reversed equations in Q3. A possible explanation is that they confused the writing of the equation with the solution process. This was not clear and needs more careful investigation.

References

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education, 13*(*1*), 16-30.

Kuchemann, D. (1978). Children's understanding of numerical variables. *Mathematics in Schools*, *7*(*4*), 23-27.

MacGregor, M., & Stacey, K. (1993). Cognitive models underlying students' formulation of simple linear equations. *Journal for Research in Mathematics Education, 24*(*3*), 217-232.

MacGregor, M., & Stacey, K. (1997). Students' understanding of Algebraic notation. *Educational Studies in Mathematics, 33***,** 1-19.

Paige, J. M., & Simon, H. A. (1966). Cognitive processes in solving algebra word problems. In B. Kleinmuntz (Ed.), *Problem solving: Research, method and theory* (pp. 51-148). Wiley, New York.